Complex Numbers Part I – Background

Let's talk about imaginary numbers.

If you learned about them, you might remember it as the day you realized that your teacher's claim of *you're going to need this math* was full of it all along. Maybe you even felt lied to; years ago you were told you can't take the square root of a negative number and if you asked why they said "**Because we said so**" so you filed it away in your list of things you know about math until suddenly you're told "*Just kidding!* You *can* do that and here's a test on it."

It might comfort you somewhat to know that when imaginary numbers first showed up most people including brilliant important mathematicians *also* thought they were full of it. That's how they got their name; Descartes was writing about solving polynomials and he would have just written about *numbers* except people wouldn't stop prattling on about how you got other solutions if you made up nonsense so he made it clear he was talking about the *real* numbers and dismissed the other numbers as *imaginary*, presumably while sneering at them over his nonsense-defying moustache.

So how did we get from that sensible position to where we are today where not only do we accept imaginary numbers, we've decided that being able to manipulate them is so important that every single person who managed to get through intro algebra *needs* to learn it? Well, the story goes back to the 16th century, when people were trying to find zeros of cubic polynomials. For what? Well, because they were curious. Bear with me.

You're probably familiar with the *quadratic formula*, which gives you a way to find the zeros of any quadratic equation whatsoever. And we know that sometimes that formula gives you negative numbers under the square root sign. But this didn't really bother people because whenever that happened, the graph of the quadratic didn't cross the x-axis – there were clearly no zeros, and sure enough the formula to find zeros just gave you numbers that don't exist, so that was alright. But a few hundred years ago some smart people found out that there's also a *cubic formula*, a method that will give you the zeros of any cubic equation. And sure enough sometimes the formula had the gall to ask you to take square roots of negative numbers. But this time something weird came up. It turned out that if you turned a blind eye to the fact that you were dealing with nonsense and just treated these absurd square roots like regular numbers, sometimes they'd cancel out along the way and what you'd get were perfectly legitimate real solutions to the equation. And this was weird. It's as if someone told you they could correctly guess anybody's age but only if they consulted their invisible unicorn – you probably wouldn't believe them until they kept getting so many right guesses that you finally caved and asked about somebody's age to the empty space where they claimed their unicorn lived and suddenly you saw the answer appear in your head clear as day. You might be forced to rethink your very conception of what it means for an animal or a number to be *real*.

So how did this canceling work? Well, suppose you'd never heard of complex numbers and you were doing perfectly ordinary algebra and you came up against $\sqrt{-3} \times \sqrt{-3}$. A less imaginative person might look at that and say "Well, I tried. Answer doesn't exist, let's pack up and go." But say you didn't take no for an answer. You want to know: if $\sqrt{-3} \times \sqrt{-3}$ were anything, what would it be? Well, to say that *this* is a square root of *that* means that *this squared* equals *that*. So if there's a square root of negative three, then whatever it is, multiplying it by another square root of negative three – squaring it – had better give you negative three, a perfectly ordinary number. I mean if it didn't, why would we even call it a square root? If you're a square root of a number and you can't even square yourself to get that number, how would you even get out of bed in the morning – what's the point? For something called a square root you're sure not very good at it. So these creative mathematicians just went and wrote $\sqrt{-3} \times \sqrt{-3} = -3$ and ran with it, while the more sensible among them shook their heads and watched with the kind of approving eyes a father gives his daughter's first boyfriend.

Now maybe you're thinking “That's stupid, you can't just make up rules so they work the way you want. I mean if you could then I could take, say, 0/1 times 2/0 and say sure you can't divide by zero but there's one on top and one on the bottom so we *should* be able to cancel them and we get 2.” And I'd say hey, you want to make that rule? You can! Except oh look, 0/1 = 0/2 since they're both zero, so let's multiply both sides by 2/0. Cancel zeros on the left, cancel zeros on the right aand... well good. One equals two. And... so does everything else. If we accept my made-up rule, we can solve equations we couldn't solve before. If we accept *your* made-up rule, then we completely break math. So my rule is interesting... and yours is kind of bad.

See, I've heard a lot of people complain that math is just a bunch of arbitrary made-up rules, and they're exactly half right. Math is just a bunch of made-up rules. But they're anything but arbitrary. Making up good rules takes imagination and curiosity. Why would anyone care about made-up rules, you ask? Well, why do you read books and watch movies? Why do get excited when Gandalf faces down the Balrog? I mean if the Tolkien had wanted he could have said “and then Gandalf sprouted giant invincible tentacles to strangle the Balrog and shot him with his all-destroying anti-Balrog eye lasers and the explosion was so great that Sauron died. The end.” and sure that's a story but you're probably not going like it because it completely threw out the rules. See the world has this internal consistency, it makes sense, and by the time we meet the Balrog we already pretty much accept what you can and can't do. Yes the heroes can still surprise us but one does not simply walk into Mordor with a nuclear bomb because we've been to led to understand that this universe doesn't work that way – and if the heroes are going to win they're going to have to do it constrained by its rules and that's what makes it interesting. It's part of what makes the story beautiful.

In math we make rules for the same reason – to see how the consequences will play out. And just like in fiction, a lot of our rules come from how we think nature works. Tolkien never specified that Frodo can't fly, but we assume he can't because... it makes sense. So what do we do when we want to make up rules but there's nothing in nature to guide us? Well, we go with what makes sense – what seems interesting, or most beautiful. We make rules based on how our souls tell us things ought to work. The incredible thing about mathematics is that those weird rules we thought were simply the most beautiful, however abstract and irrelevant and useless – time and time again, in our adventures through the world around us, we find them. Imaginary sculptures we built out of dreams and reasons, and there they are in nature, looking back at us, asking us “I've been here the whole time; what took you so long?

So yes, imaginary numbers are imaginary. Just like all the other numbers. So let's start with that. In the next video in this series, we're going to make up rules for made-up numbers and see what happens. Not because we think it won't be useful or because we want to solve some specific problems, but for the same reason you might make up rules for how to carry out a fight between a Charizard and Batman – out of pure curiosity, out of interest, just for fun. And because sometimes, the more you think about one little idea – like the square root of a negative number – the more you'll see it connect, the more the ideas will dance and grow before your eyes until you're sure you're following a trail that was left by something so much bigger than yourself.